WORKING FALL SEMINAR: THE WORK OF GROMOV AND MCDUFF ON SYMPLECTIC 4-MANIFOLDS

1. Overview

The goal of this seminar is to get acquainted with the essential techniques of *closed* pseudo-holomorphic curves in *closed* symplectic 4-manifolds and to understand how these techniques are used to prove the foundational theorems of symplectic topology due to Gromov [1] and McDuff [2], from a somewhat more modern point of view.

We will follow Chris Wendl's lecture notes [5], as they provide a very straightforward path to the basic hard theorems of symplectic geometry, namely: recognizing $(\mathbb{C}P^2, \omega_{FS})$ and $(S^2 \times S^2, \sigma \oplus \sigma)$ based on their rational curves[1]. The key technical input we will work towards is understanding why exceptional curves cannot break under deformation. This is the main ingredient in many other extraordinary 4-dimensional phenomena, most notably in quantitative symplectic geometry, for instance in the study of symplectic ball packings and in showing that symplectic embedding problems can behave drastically differently from their volume-preserving counterparts.

Following the structure of [5], we will assume basic knowledge of symplectic geometry and we will review the basic theory of pseudoholomorphic curves but keep most of the analysis in the background. Hopefully, the material covered will not only provide the necessary background for more advanced research topics in symplectic 4-manifolds but also provide a lot of "moral" motivation for many techniques in other areas of symplectic topology, such as Floer theory and SFT.

Being somewhat optimistic[2] the logical follow-up would be a Working Spring Seminar on SFT covering the second part of [5] accompanied with Wendl's other wonderful book [6]. There, we would see how to adapt the techniques of compact holomorphic curves to develop a theory of *punctured* holomorphic curves on *symplectic cobordisms*.

2. Tentative schedule

Session 1 ([5, Ch. 1, p. 1-15]): Basic facts and examples of symplectic 4-manifolds. Overview of the theorems that will be covered.

[1] We will cover more or less ground similar to Chapter 9.4 of the McDuff–Salamon bible [4]

[2] "*Pessimismo dell'intelligenza, ottimismo della volontà*" as A. Gramsci would write.

Session 2 ([5, Ch. 2, p. 17-32, up to 2.1.4/Ch. 2, p. 45-51]): Basic notions of (compact) pseudo-holomorphic curves. Getting acquainted with *automatic transversality* in dimension 4 (Theorem 2.44).

Session 3 ([5, Ch. 3, p. 53-68]): Discuss the complex blow-up and how to perform it in the symplectic category (mostly moral, no technical proofs). Introduce basic notions of Lefschetz fibrations/pencils (only 3.3).

Session 4 ([5, Ch. 4, p. 79-87, up to 4.3]): Understand how the moduli space of smooth, embedded rational curves gets compactified and the importance of the "codimension 2 boundary". If there is time, sketch also 4.4.

Session 5 ([5, Ch. 5, p. 93-96]): Show that (-1)-curves cannot break under deformation and explain the symplectic analogue of Castelnuovo's criterion. Deduce that blowing down a maximal collection of pairwise disjoint symplectic exceptional spheres produces a minimal symplectic manifold. **Bonus:** One could give an overview of McDuff-Polterovich [3] which is the cornerstone paper of symplectic packings and relies crucially on Theorem B.

Session 6 ([5, Ch. 6.1, p. 97-100]): Show Theorems F and G: a symplectically embedded sphere of nonnegative self-intersection implies the existence of a Lefschetz pencil (and its parametric version).

Session 7 ([5, Ch. 6.2, p. 100-103]): Prove Theorems A, D, and E, culminating in the rational/ruled classification: a closed symplectic 4-manifold containing a symplectically embedded sphere of nonnegative self-intersection is rational or ruled. In the minimal case, this gives the corresponding recognition results for symplectic ruled surfaces, $\mathbb{C}P^2$, and $S^2 \times S^2$.

More...? We can see how the above results can be applied to further questions: fillings of contact manifolds, the topology of symplectomorphism groups, more complicated embedding problems. Alternatively, we can discuss a generalization of Theorem A to higher-genus symplectic surfaces and its relation to Taubes' SW = Gr theorem, which relates Seiberg–Witten invariants to Gromov invariants.

Sessions $2, 3$, and 4 cover a big portion of the book but mostly collect and organize key notions and results. Sessions $5, 6, 7$ are our goals and synthesize most of what we will have learned by then and therefore will be slightly more challenging.

References

[1] M. Gromov. "Pseudo holomorphic curves in symplectic manifolds". In: *Inventiones mathematicae* 82 (1985), pp. 307–347.

[2] D. McDuff. "The structure of rational and ruled symplectic 4-manifolds". English. In: *J. Am. Math. Soc.* 3.3 (1990), pp. 679–712. DOI: `10.2307/1990934`.

[3] D. McDuff and L. Polterovich. "Symplectic packings and algebraic geometry". English. In: *Invent. Math.* 115.3 (1994), pp. 405–429. DOI: `10.1007/BF01231766`.

[4] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*. Vol. 52. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004, pp. xii+669. DOI: `10.1090/coll/052`.

[5] C. Wendl. *Holomorphic curves in low dimensions. From symplectic ruled surfaces to planar contact manifolds*. English. Vol. 2216. Lect. Notes Math. Berlin: Springer, 2018. DOI: 10.1007/978-3-319-91371-1.

[6] C. Wendl. *Lectures on contact 3-manifolds, holomorphic curves and intersection theory.* English. Vol. 220. Camb. Tracts Math. Cambridge: Cambridge University Press, 2020. DOI: 10.1017/9781108608954.